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Short Communication

What are the repeated frequencies?

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1. Introduction

Craig [1] said when two frequency values differ by only 1% or so, they could be regarded as a double repeated frequency since a precisely repeated frequency may be an extremely unlikely accident in practice. Such a conception may be based on the precision of the measurement in engineering environments, and has been widely accepted by designers and engineers for years. At times, the original distinct natural frequencies may be measured with identical values due to the precision of measuring apparatus. On the other hand, the repeated frequencies evaluated analytically may be measured with different values because of the deficiencies in the component manufacturing, structural assembling as well as the uncertainties of material properties. In most cases, structural dynamic analyses for any but the smallest systems are implemented numerically with the finite element method (FEM). Repeated frequencies may be found with a slight difference due to the computational precision in computers. Therefore, it would be beneficial to designers and engineers to investigate the effect of such a small discrepancy in frequencies on, for instance, their respective derivatives. This is because the procedures for derivative computations of distinct and repeated frequencies are completely different. A small tolerance of two frequencies may sometimes incur a large discrepancy between their derivatives. Besides, it is increasingly important

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to employ frequency derivatives in both gradient-based structural optimization algorithms and dynamic system identifications, where accurate evaluations of the derivatives are absolutely required.

So far, little work has been done in this aspect. For this reason, this paper attempts to provide an insight into the tolerance of two closely spaced but unequal natural frequencies when they are treated indiscriminately as double repeated frequencies. Efforts will be made to give rise to a reasonable criterion of 0.1% based on the first-order derivative for the coalescence of closely spaced frequencies, so that the discrepancy of the derivatives may lie within an acceptable limit in the general case.

2. Derivative calculation

Provided that there exist two distinct eigenpairs (ω_1, ϕ_1) and (ω_2, ϕ_2) in a vibration system, and the related mode shapes have been orthogonally normalized with respect to the mass matrix. The frequency values are spaced so closely to each other that

$$\omega_2 - \omega_1 \leqslant c\omega_1, \quad 0 < c << 1, \tag{1}$$

that is,

$$\omega_1 < \omega_2 \leqslant (1 + c)\omega_1. \tag{2}$$

Herein, let us take the extreme value

$$\omega_2 = (1 + c)\omega_1. \tag{3}$$

According to Craig's suggestion, c takes 1%. The following expression can be obtained by ignoring the higher order term of small value:

$$\omega_2^2 = (1 + c)^2\omega_1^2 \approx (1 + 2c)\omega_1^2. \tag{4}$$

Suppose these two unequal frequencies are regarded as 'double repeated frequencies' by neglecting their discrepancy, i.e., assume $\omega_1 = \omega_2$. Such an assertion would inevitably introduce errors to their derivatives. As we know, repeated frequencies are not differentiable in the common sense, i.e., the Frechet derivative does not exist. Only directional derivatives can be found in the design space [2]. The computation of derivatives for repeated frequencies can be achieved by solving a sub-eigenvalues problem [2,3].

For an undamped vibration system, the governing eigenvalue equation is

$$([K] - \omega_i^2[M])\{\phi\}_i = 0, \quad i = 1,\ 2, \tag{5}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices of the system, respectively.

Let us define a vibration mode according to the dynamic theorem

$$\{\tilde{\phi}\} = [\{\phi\}_1, \{\phi\}_2]\{C\}, \tag{6}$$

where $C=[c_1,\ c_2]^{\mathrm{T}}$ is a vector of constants. The condition of $[M]$-orthonormalization of $\{\tilde{\phi}\}$ requires

$$\{C\}^{\mathrm{T}} \cdot \{C\} = 1. \tag{7}$$

Substituting Eq. (6) into Eq. (5) and then differentiating it with respect to the design parameter x yields

$$\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}-\frac{\partial\omega_1^2}{\partial x}[M]\right)\{\tilde{\phi}\}+([K]-\omega_1^2[M])\frac{\partial\{\tilde{\phi}\}}{\partial x}=0. \tag{8}$$

Premultiplying both sides of Eq. (8) by $[\{\phi\}_1,\{\phi\}_2]^{\mathrm{T}}$ yields

$$[G][C]=\frac{\partial\omega_1^2}{\partial x}\{C\}, \tag{9}$$

where $[G]$ is composed of

$$[G]=\begin{bmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix} \tag{10}$$

in which

$$g_{mn}=\{\phi\}_{\mathrm{n}}^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_m,\quad m,n=1,2. \tag{11}$$

In fact, those two frequencies are distinct. The exact derivative of ω_1^2 is calculated by

$$\frac{\partial\omega_1^2}{\partial x}=\{\phi\}_1^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_1=g_{11}. \tag{12}$$

And the exact derivative of ω_2^2 is

$$\begin{aligned}\frac{\partial\omega_2^2}{\partial x}&=\{\phi\}_2^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_2^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_2\\ &\approx\{\phi\}_2^{T}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_2-2c\omega_1^2\{\phi\}_2^{T}\frac{\partial[M]}{\partial x}\{\phi\}_2\\ &=g_{22}-2c\omega_1^2\{\phi\}_2^{T}\frac{\partial[M]}{\partial x}\{\phi\}_2.\end{aligned} \tag{13}$$

Moreover, following relations hold due to the orthogonality of modes:

$$\begin{gathered}\{\phi\}_1^{\mathrm{T}}[M]\{\phi\}_2=0,\\ \{\phi\}_1^{\mathrm{T}}[K]\{\phi\}_2=0.\end{gathered} \tag{14}$$

Differentiating Eq. (14) with respect to the design parameter x yields

$$\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}[M]\{\phi\}_2+\{\phi\}_1^{\mathrm{T}}\frac{\partial[M]}{\partial x}\{\phi\}_2+\{\phi\}_1^{\mathrm{T}}[M]\frac{\partial\{\phi\}_2}{\partial x}=0, \tag{15}$$

$$\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}[K]\{\phi\}_2+\{\phi\}_1^{\mathrm{T}}\frac{\partial[K]}{\partial x}\{\phi\}_2+\{\phi\}_1^{\mathrm{T}}[K]\frac{\partial\{\phi\}_2}{\partial x}=0. \tag{16}$$

Note that the above operation is valid only for distinct frequencies. By performing the manipulation of Eq. (16)−Eq. (15) $\times\,\omega_1^2$, one gets

$$\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}([K]-\omega_1^2[M])\{\phi\}_2+\{\phi\}_1^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_2 + \{\phi\}_1^{\mathrm{T}}([K]-\omega_1^2[M])\frac{\partial\{\phi\}_2}{\partial x}=0. \tag{17}$$

In the above equation, the third term on the left side vanishes due to the symmetry of the global stiffness and mass matrices. But the first term does not, owing to the tiny discrepancy between ω_1 and ω_2. Hence, the off-diagonal terms of $[G]$ is rewritten as

$$\begin{aligned} g_{12}&=\{\phi\}_1^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_2=-\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}([K]-\omega_1^2[M])\{\phi\}_2\\ &=-\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}(\omega_2^2-\omega_1^2)[M]\{\phi\}_2\approx-2c\omega_1^2\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}[M]\{\phi\}_2. \end{aligned} \tag{18}$$

In the derivation of the above equation, the fundamental equation (5) has been used. Similarly, one could get

$$g_{21}=\{\phi\}_2^{\mathrm{T}}\left(\frac{\partial[K]}{\partial x}-\omega_1^2\frac{\partial[M]}{\partial x}\right)\{\phi\}_1=-\{\phi\}_2^{\mathrm{T}}([K]-\omega_1^2[M])\frac{\partial\{\phi\}_1}{\partial x}=g_{12}. \tag{19}$$

Consequently, it is recognized that $[G]$ is a symmetric matrix and all its eigenvalues are real numbers.

Generally, we can obtain the following relations for the eigenvalues of $[G]$ according to the Gerschgorin theorem about the matrix eigenvalues [4]

$$|\lambda_1-g_{11}|\leqslant|g_{12}|, \tag{20a}$$

$$|\lambda_2-g_{22}|\leqslant|g_{12}|, \tag{20b}$$

where λ_1 and λ_2 are the eigenvalues of $[G]$ and are assumed $\lambda_1\leqslant\lambda_2$. If $g_{12}=0$, both eigenvalues would take the diagonal terms of $[G]$, respectively.

As is stated in Eq. (9), the eigenvalues of $[G]$ are the directional derivatives for the 'double repeated frequencies' [2]. Therefore, from Eq. (20a) in conjunction with Eqs. (12) and (18), one gets

$$\left|\lambda_1-\frac{\partial\omega_1^2}{\partial x}\right|\leqslant 2c\omega_1^2\left|\frac{\partial\{\phi\}_1^{\mathrm{T}}}{\partial x}[M]\{\phi\}_2\right|. \tag{21}$$

In addition, we could get

$$\left|\lambda_2-g_{22}+2c\omega_1^2\{\phi\}_2^{\mathrm{T}}\frac{\partial[M]}{\partial x}\{\phi\}_2\right|\leqslant\left|\lambda_2-g_{22}\right|+\left|2c\omega_1^2\{\phi\}_2^{\mathrm{T}}\frac{\partial[M]}{\partial x}\{\phi\}_2\right|. \tag{22}$$

According to Eq. (20b) in conjunction with Eqs. (13) and (18), one gets

$$\left|\lambda_2 - \frac{\partial \omega_2^2}{\partial x}\right| \leqslant 2c\omega_1^2 \left(\left| \frac{\partial \{\phi\}_1^{\mathrm{T}}}{\partial x} [M] \{\phi\}_2 \right| + \left| \{\phi\}_2^{\mathrm{T}} \frac{\partial [M]}{\partial x} \{\phi\}_2 \right| \right). \tag{23}$$

Due to the fact that the derivative of a mode can be expanded in the vibration mode space by Fox's method [5], the common term on the right sides of Eqs. (21) and (23) may become relatively small or zero according to the mode orthogonality. However, much attention should be paid to the second term in Eq. (23). Occasionally, it may contribute greatly to the error of the frequency derivative and worsen its accuracy, even though the derivative of the mass matrix only has an influence at the element level. We will illustrate in the following examples that the error of the frequency derivatives may become much larger than that of the frequencies. And the accuracy of the derivatives can only be ensured when the discrepancy of frequencies is controlled within the tolerance of 0.1%.

3. Numerical examples

Two schematic problems are used to illustrate the tolerances of the frequencies and their corresponding derivatives. In the first example, the criterion of 1% is acceptable. Nevertheless, in the second example, the discrepancy of the derivatives is much greater than that of the frequencies. To ensure the correctness of the frequency derivative, at least 0.1% has to be imposed on the frequency tolerance.

Example 1 (*Two-bar planar truss*). A two-bar planar truss is shown in Fig. 1. Suppose that the material properties and cross-sectional areas are the same for both bars. α is the only design variable with which nodes 2 and 3 can be shifted symmetrically in the vertical direction to raise the fundamental frequency. The reduced global stiffness and mass matrices are, respectively,

$$[K] = \frac{2AE}{D} \begin{bmatrix} \cos^3 \alpha & 0 \\ 0 & \sin^2 \alpha \cos \alpha \end{bmatrix},$$

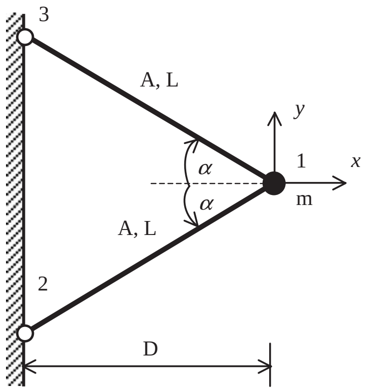

Fig. 1. Two-bar planar truss structure.

$$[M]=\begin{bmatrix}\frac{2AD\rho}{3\cos\,\alpha}+m & 0\\ 0 & \frac{2AD\rho}{3\cos\,\alpha}+m\end{bmatrix}.$$

Hence the two frequencies are

$$\omega_x^2=\frac{2AE\cos^4\alpha}{D(2AD\rho/3+m\,\cos\,\alpha)}\quad\text{and}\quad\omega_y^2=\frac{2AE\sin^2\alpha\cos^2\alpha}{D(2AD\rho/3+m\,\cos\,\alpha)}$$

and

$$\frac{\omega_y}{\omega_x}=\sqrt{\frac{\omega_y^2}{\omega_x^2}}=\mathrm{tg}\alpha.$$

When $\alpha=45^\circ$, the exact double repeated frequencies come forth, and the fundamental frequency reaches its maximum

$$\omega_1^2\big|_{\max}=\omega_x^2=\omega_y^2=\frac{AE}{D(4AD\rho/3+\sqrt{2}m)}.$$

The derivatives of the two frequencies are, respectively,

$$\frac{\mathrm{d}\omega_x^2}{\mathrm{d}\alpha}=\frac{-2AE\cos^3\alpha\,\sin\,\alpha(8AD\rho/3+3m\,\cos\,\alpha)}{D(2AD\rho/3+m\,\cos\,\alpha)^2},\tag{24}$$

$$\frac{\mathrm{d}\omega_y^2}{\mathrm{d}\alpha}=\frac{2AEm\cos^2\alpha\sin^3\alpha}{D(2AD\rho/3+m\,\cos\,\alpha)^2}+\frac{4AE(\sin\,\alpha\cos^3\alpha-\cos\,\alpha\sin^3\alpha)}{D(2AD\rho/3+m\,\cos\,\alpha)}.\tag{25}$$

Then, the analytical derivatives of the repeated frequencies are

$$\left.\frac{\mathrm{d}\omega_x^2}{\mathrm{d}\alpha}\right|_{45^\circ}=\frac{-\left(\frac{8\sqrt{2}}{3}AD\rho+3m\right)AE}{2\sqrt{2}D(2AD\rho/3+m/\sqrt{2})^2},$$

$$\left.\frac{\mathrm{d}\omega_y^2}{\mathrm{d}\alpha}\right|_{45^\circ}=\frac{AEm}{2\sqrt{2}D(2AD\rho/3+m/\sqrt{2})^2}.$$

Due to the deficiencies in structural fabricating and its assembling, the design parameter α may not reach 45° exactly. Then, computational errors are brought into the frequencies and their derivatives, respectively. Table 1 compares the obtained results under the assumption of $A=E=D=m=\rho=1$.

In this example, $g_{12}=0$ before exactly repeated frequencies emerging. From Table 1, it is observed that the derivatives of ω_y^2 (the lower frequency) are the same with the two approaches. Errors exist only for the derivatives of ω_x^2 (the higher one), and are not greater than those of the frequencies.

Table 1
Error comparisons of the frequencies with their derivatives

Design variable	$\dfrac{\omega_y}{\omega_x}$	Frequency error (%) $\left(1-\dfrac{\omega_y}{\omega_x}\right)$	As distinct with Eqs. (24), (25) $\dfrac{\partial\omega_y^2}{\partial a}$	As distinct with Eqs. (24), (25) $\dfrac{\partial\omega_x^2}{\partial a}$	As repeated with Eq. (9) $\dfrac{\partial\omega_y^2}{\partial a}$	As repeated with Eq. (9) $\dfrac{\partial\omega_x^2}{\partial a}$	Error (%) $\dfrac{\partial\omega_y^2}{\partial a}$	Error (%) $\dfrac{\partial\omega_x^2}{\partial a}$
44°	0.966	3.4	0.231	−1.299	0.231	−1.287	0.0	0.93
44.5°	0.983	1.7	0.209	−1.284	0.209	−1.278	0.0	0.47
44.75°	0.991	0.9	0.198	−1.276	0.198	−1.273	0.0	0.24
45°	1.000	0.0	0.187	−1.269	0.187	−1.269	0.0	0.0

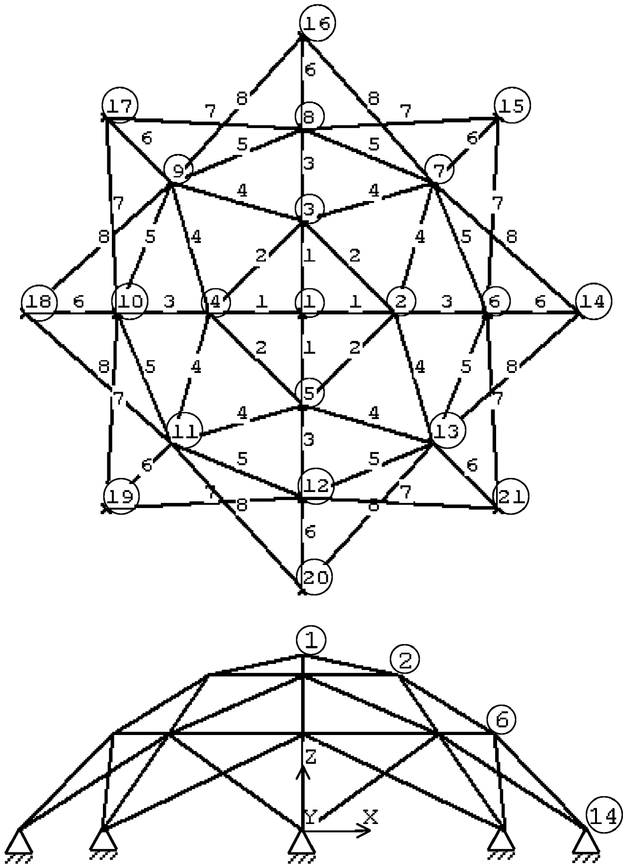

Fig. 2. Dome structure.

Example 2 (*Dome structure*). The dome structure [2] shown in Fig. 2 is investigated for derivatives of the fundamental frequency with respect to the size parameter. Fifty-two bars are linked into eight groups with all the cross-section areas being 10 cm^2. The dome is symmetric with respect to

Table 2
Representative node coordinates of the dome structure

Node	Coordinates (m)		
	x	y	z
1	0.0	0.0	9.25
2	5.0	0.0	8.22
6	10.0	0.0	5.14
14	15.0	0.0	0.0

Table 3
Error comparisons of the frequencies with their derivatives

Abscissa of Node 14 (m)	ω_1 (rad/s)	ω_2 (rad/s)	Error (%)	As distinct with Eqs. (12), (13)		As repeated with Eq. (9)		Error	
				$\frac{\partial \omega_1^2}{\partial s}$	$\frac{\partial \omega_2^2}{\partial s}$	$\frac{\partial \omega_1^2}{\partial s}$	$\frac{\partial \omega_2^2}{\partial s}$	$\frac{\partial \omega_1^2}{\partial s}$	$\frac{\partial \omega_2^2}{\partial s}$
				$(\times 10^5)$		$(\times 10^5)$		(%)	
14.5	178.49	180.45	1.1	−4.224	−7.522	−4.224	−6.224	0.0	17.3
14.7	178.59	179.88	0.7	−4.822	−6.371	−4.822	−5.521	0.0	13.3
14.9	178.67	179.15	0.3	−5.449	−5.787	−5.449	−5.477	0.0	5.4
15.0	178.69	178.69	0.0	−5.787	−5.787	−5.787	−5.787	0.0	0.0

x- and y-axis. Coordinates of the representative nodes are listed in Table 2. Let Young's modulus $E = 210\,\text{GPa}$ and material density $\rho = 7850\,\text{kg/m}^3$. It is worthwhile noting that the fundamental frequency is a double repeated one. In this example, structural dynamic analysis is implemented by the FEM. Suppose that we produce a perturbation on the abscissa of Node 14. The derivatives of the first two frequencies with respect to the size Group 5 are documented in Table 3 for comparison.

A simple look at Table 3 reveals obviously that the derivative accuracy is rather poor in comparison with that of the frequencies. The derivative discrepancies are at least an order of magnitude larger than those of the frequencies. For instance, when the error of the frequencies is 1.1%, the discrepancy of the derivatives could reach 17.3%. Even the error of the frequencies is reduced to 0.3%, the derivative discrepancy still remains 5.4%. Though the off-diagonal term g_{12} vanishes in this example, the derivative value of the global mass matrix is much greater than itself, which inevitably leads to the product of $\left|\phi_2^{\mathrm{T}} \frac{\partial M}{\partial x} \phi_2\right|$ fairly large. Therefore, to ensure a reasonable tolerance of the frequency derivatives within, for instance, 3–5%, a tolerance of 0.1% or so has to be imposed on the difference of two frequencies with unequal values. Only in this case, could they be treated as double repeated frequencies.

4. Conclusion

In this paper, the tolerance of two closely spaced but unequal frequencies is investigated when they are regarded as double repeated ones since this circumstance is quite often encountered in practical engineering systems. On the basis of the tolerance of their respective derivatives calculated with appropriate approaches, it is found that when two frequencies separate by 1% or so, the error of the frequency derivatives may become significant. A tolerance level of 0.1% is suggested for the frequency values according to the tolerance of their corresponding derivatives.

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